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Fracture detection problems: applications and limitations of the energy momentum tensor and related invariants

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Abstract

We show that under certain circumstances, if displacement measurements are made inside and/or outside a body, it is possible to use two invariants based on the energy momentum tensor to determine (to some extent) the crack direction and length for cracks in bidimensional problems or the crack direction and area for cracks in three dimensional problems. This is done for a certain family of non-linear materials with a given toughness which includes linearly elastic materials with quadratic strain energy and power law elastic. One of the limitations is that the crack must be straight in 2D or planar in 3D.

1. Introduction

In hydraulic fracture, a proppant-laden viscous fluid is injected into a wellbore and pressurized such as to initiate the fracture of the surrounded rock (Veatch, 1989). This technique has several engineering applications, mainly to the stimulation of oilfield reservoirs. To this extent, it becomes a useful tool if the fracture can somehow be located and directed towards the oilfield reservoir. The displacements of the wellbore wall as the fluid pressure is being applied can be measured by means of a caliper tool. There are also other devices for measuring the deformation of a borehole wall. For example, Ito and Hayashi (1996) have introduced a new device based on electrical resistance strain gauges. This device attached to the surface of a packer element deforms with the borehole wall when the packer is pressed against it. It is also possible to impede the fluid flow into the crack by the so-called technique of sleeve fracturing. The case where the fluid is allowed to infiltrate the crack and assumed to exert a constant pressure on the crack faces can also be considered with some modifications. The problem becomes that of determining the location and extension of a stress free crack by using the knowledge of the displacements and stresses around

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the wellbore. This is an inverse problem whose antiplane counterpart has been studied previously (Atkinson and Aparicio, 1994, Aparicio and Pidcock, 1996). The idea that we put forward in this paper is based on the use of the energy momentum tensor and related invariants to determine the inclination and length of a straight stress free crack in a two dimensional body and the inclination and area of a planar stress free crack in a three-dimensional body. In this problem, it is assumed that the material toughness is known, displacements and stresses are known around the wellbore and differentiation of the displacements around the wellbore are permitted (it may be possible to avoid needing to differentiate the displacements by using a measuring technique such as that described above). However, the material can belong to a certain family of non-linear materials which includes some of a power law type.

We note that the derivation given below is a small strain theory but with a non-linear stress strain law. Physically non-linear constitutive relations and small strains are appropriate for rocks (see, e.g., Jaeger and Cook, 1979 and Fjaer et al. 1992) and the deformation theory of plasticity. It is possible that a method similar to the one outlined below could be applied to large strain situations to some extent, i.e., the J integral formulation is also valid for large strains and any energy density. However, we have not investigated this in detail.

The path independent integrals that are related to the energy momentum tensor can be deduced from Noether's theorem (Noether, 1918, Knowles and Sternberg, 1972) which relates symmetry groups of a variational problem to conservation laws of the associated Euler–Lagrange equations. It is therefore possible to construct similar path independent integrals from any given variational problem. The problem of hydraulic fracturing where the fluid is allowed to flow inside the crack is rather complicated, it involves the interaction of the surrounded rock, frequently modeled as linearly elastic or poroelastic (Atkinson and Craster, 1991) and a non-Newtonian fluid which is normally assumed to be of a power law type. There are also additional effects caused by fluid leakoff through the fracture walls when the material is assumed to be permeable. However, variational formulations have been constructed that take into account all these interactions (Biot et al., 1986, Advani et al., 1992).

Although the results presented in this work are mainly oriented towards the problem of hydraulic fracturing, we point out that they can be equally useful in other kinds of problems with similar requirements like pressure vessels, etc.

2. Path independent integrals for non-linear materials

In this section we establish the general formulation that will allow us to determine the angle of inclination and length of a straight stress free crack in a two-dimensional problem or the inclination and area of a planar stress free crack in a three-dimensional problem. The path independent integrals we will use are normally known as the J (or F) integral and the M integral (Eshelby, 1970 and 1975). The J integral is already known to be path independent in non-linear elastic materials; we only review here its derivation due to Eshelby (1975). The M integral is believed to be path independent in materials that have a Lagrangian which is a homogeneous function of some degree in the strain components (Budiansky and Rice, 1973).

Let us assume that the Lagrangian $\mathscr{L}$ of a non-linear elastic material is a function of the strain tensor only ε_{ij} of the form

$$\mathscr{L} = f(u, v),$$

where

$$u = C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}, \quad v = D_{ij}\varepsilon_{ij},$$

and C_{ijkl}, D_{ij} are constant tensors.

The stress tensor σ_{ij} is defined as

$$\sigma_{ij} = \frac{\partial \mathscr{L}}{\partial \varepsilon_{ij}}, \tag{1}$$

thus if u_i is the displacement field which is related to the strain tensor by $\varepsilon_{ij} = (u_{i,j} + u_{j,i})/2$, then the Euler equations transform themselves into

$$-\frac{\partial \mathscr{L}}{\partial u_i} + \left(\frac{\partial \mathscr{L}}{\partial u_{i,j}}\right)_{,j} = \left(\frac{\partial \mathscr{L}}{\partial \varepsilon_{ij}}\right)_{,j} = \sigma_{ij,j} = 0,$$

which are the equilibrium equations.

To be able to construct an M integral that is path independent for some non-linear constitutive laws, we need that, for a given constant c, the following relationship holds

$$c\mathscr{L} = \tfrac{1}{2}\sigma_{ij}\varepsilon_{ij}. \tag{2}$$

The definition of the M integral is stated later in this section. Substituting (1) into the above eqn, we reach the conclusion that the function f must satisfy

$$cf(u, v) = u\frac{\partial f}{\partial u} + \frac{1}{2}v\frac{\partial f}{\partial v},$$

whose general solution is

$$F\left(\frac{f(u, v)}{u^c}, \frac{v^2}{u}\right) = 0. \tag{3}$$

If $c = 1$ and $F(x, y) = x - 1/2$, eqn (3) will give us the Lagrangian of a linear anisotropic elastic material

$$\mathscr{L} = f(u, v) = \tfrac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}.$$

On the other hand, if $C_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2$, $D_{ij} = \delta_{ij}$, $c = \alpha + 1$ and $F(x, y) = x - G_o/(\alpha + 1) - \lambda_o y/2$, we have the following Lagrangian

$$\mathscr{L} = \Gamma^{\alpha}\left(\frac{G_o\Gamma}{\alpha + 1} + \frac{\lambda_o\varepsilon_{kk}^2}{2}\right)$$

where $\Gamma = \varepsilon_{ij}\varepsilon_{ij}$, which corresponds to the following non-linear constitutive law

$$\sigma_{ij} = \left(2G + \frac{\alpha\lambda\varepsilon_{kk}^2}{\Gamma}\right)\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij},$$

where $G = G_o\Gamma^\alpha$ and $\lambda = \lambda_o\Gamma^\alpha$ are variable elastic moduli. For $\alpha = 0$, the above eqn transforms itself into the constitutive law of a linear isotropic elastic material.

We define the energy momentum-tensor as

$$P_{ij} = \mathscr{L}\delta_{ij} - \sigma_{kj}u_{k,i}.$$

It can be easily proved that $P_{ij,j} = 0$ and therefore the following integral is path independent

$$J_i = \int_\Gamma P_{ij}n_j d\Gamma.$$

In two dimensions, the J_1 integral will vanish along a horizontal line where $n_1 = \sigma_{12} = \sigma_{22} = 0$ and the J_2 integral will vanish along a vertical line where $n_2 = \sigma_{11} = \sigma_{12} = 0$. In three dimensions, the J_1 integral will vanish on stress free planes orthogonal to the x_2 or x_3 axes, the J_2 integral will vanish on stress free planes orthogonal to the x_1 or x_3 axes and the J_3 integral will vanish on stress free planes orthogonal to the x_1 or x_2 axes.

The M integral is defined as

$$M = \int_\Gamma \left[x_i P_{ij} - \left(\frac{N-2c}{2c}\right)\sigma_{ij}u_i \right] n_j d\Gamma,$$

where N is the number of dimensions (2 or 3) and c the constant introduced in (2). It is a straightforward procedure to prove that

$$\left[x_i P_{ij} - \left(\frac{N-2c}{2c}\right)\sigma_{ij}u_i \right]_{,j} = 0$$

by taking into account the relationship (2).

In two dimensions, the M integral vanishes along stress free lines $x_1 = 0$ or $x_2 = 0$ (the coordinate axes). In three dimensions, the M integral vanishes on stress free planes $x_1 = 0$, $x_2 = 0$ or $x_3 = 0$.

3. Determination of the stress tensor on the boundary

The technique that we are proposing in this paper assumes the knowledge of both displacements and tractions on the (known) boundary to allow us to determine certain characteristics of the stress free crack. In the case of hydraulic (sleeve) fracturing of a wellbore, the displacements can be measured by means of a caliper tool, as was already discussed in the Introduction, and the normal traction is equal to the pressure of the proppant fluid. In this section, we briefly comment on the determination of the whole stress tensor on the boundary using the data previously mentioned.

Let us consider a point on the boundary and define a system of reference where the z axis is normal to the boundary and the x and y axes are tangent to the boundary at the point we are studying. Since the tractions are known at that point, the components σ_{zz}, σ_{yz} and σ_{xz} of the stress tensor are known, for example, in the case of hydraulic fracturing, $\sigma_{zz} = -p$ and $\sigma_{xz} = \sigma_{yz} = 0$, where p is the pressure of the proppant fluid. On the other hand, we also know the displacements around the boundary and we have established that they must be determined with sufficient accuracy as to allow us to differentiate them numerically. This means that we know all the derivatives $\partial u_i/\partial x$ and $\partial u_i/\partial y$ where $i = x, y, z$.

Equation (1) allows us to write the components of the stress tensor in terms of the derivatives of the displacements and the elastic constants only. Since we know the values of three of these stress tensor components, we have enough equations to determine the three unknown derivatives of the displacements ($\partial u_i/\partial z$) and therefore we can determine the remaining components of the stress tensor once these derivatives are known.

4. Two-dimensional crack detection

It is well known (Eshelby, 1970) that the work done by the material against the cohesive forces near the crack tip to advance the crack a quantity δu is equal to $J_i(C_\varepsilon)\delta u_i$ as $\varepsilon \to 0$, where $J_i(C_\varepsilon)$ is the J_i integral evaluated with the normal pointing outwards around a circular curve C_ε of radius ε and centered at the crack tip. If m is a unit vector tangent to the crack at its tip and pointing to the direction of advance, then, as $\varepsilon \to 0$, the number $J_i(C_\varepsilon)m_i$ will give us the work done by the material against the cohesive forces near the crack tip per unit of crack advance or *energy release rate*. For the crack to propagate, the energy release rate must be equal to some critical energy release rate G_{cr} which is related to the material toughness, a material property which is assumed to be isotropic and known in this work. In summary, we have

$$\lim_{\varepsilon \to 0} J_i(C_\varepsilon)m_i = G_{cr}.$$

This implies that J_i is of the form

$$\lim_{\varepsilon \to 0} J_i(C_\varepsilon) = G_{ij}m_j. \tag{4}$$

The tensor G_{ij} is given by

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} G_{cr} & G' \\ -G' & G_{cr} \end{bmatrix},$$

where G' is some fixed value. This holds in any cartesian system of coordinates.

In a linearly elastic isotropic material, the components of the energy release rate tensor G_{ij} and the stress intensity factors for modes I and II (K_I and K_{II}) are related as

$$G_{11} = G_{22} = \frac{\lambda + 2G}{4G(\lambda + G)}(K_I^2 + K_{II}^2)$$

$$G_{12} = -G_{21} = \frac{\lambda + 2G}{4G(\lambda + G)}(2K_I K_{II}).$$

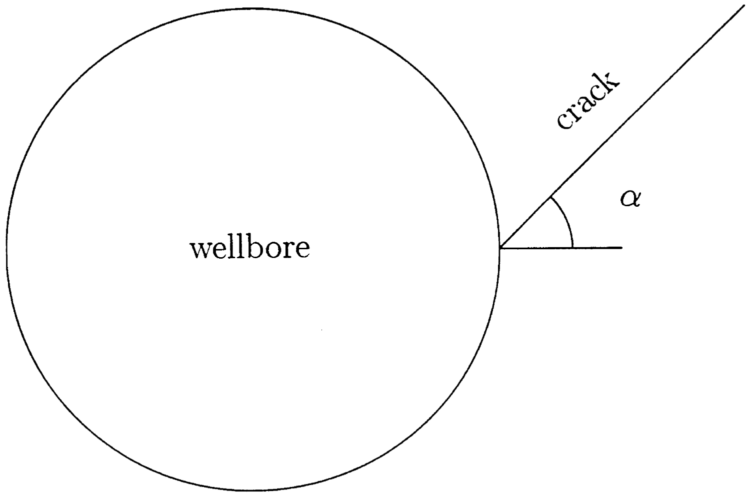

Fig. 1. Straight bidimensional crack coming from the cross section of a wellbore at an angle α with respect to the horizontal.

In Fig. 1, a stress free crack inclined an angle α with respect to the horizontal is being propagated from a wellbore. In a cartesian system of coordinates y_1y_2 where y_1 is directed along the crack, we will have that J_1^y vanishes along the crack and gives us the critical energy release rate G_{cr} when the integral is carried out with the normal pointing outwards along a small circle that surrounds the crack tip. Since J_1^y is path-independent, we have

$$G_{cr} = \int_H P_{1j}^y n_j^y ds = J_1^y(H), \tag{5}$$

where the integral is carried out with the normal pointing inwards around the wellbore H ($J_1^y(H)$ means the integral J_1^y along H). We are assuming that there are no stresses at infinity. By expressing the involved quantities in the horizontal-vertical cartesian system of reference x_1x_2, we have

$$J_1^y(H) = J_1^x(H)\cos\alpha + J_2^x(H)\sin\alpha.$$

Substituting the above eqn into (5) we get

$$G_{cr} = J_1^x(H)\cos\alpha + J_2^x(H)\sin\alpha. \tag{6}$$

Notice that although the integrals J_i^x do not necessarily vanish along an inclined stress free crack, eqn (6) is still valid.

The crack's inclination can be determined from eqn (6) as

$$\alpha = \arcsin\left(\frac{G_{cr}}{\sqrt{J_1^x(H)^2 + J_2^x(H)^2}}\right) - \arctan\left(\frac{J_1^x(H)}{J_2^x(H)}\right). \tag{7}$$

In order to determine the crack's length, we study the value of the M integral as it is carried out with the normal pointing outwards around a small circle C_ε of radius ε and centered at the crack tip.

$$M(C_\varepsilon) = \int_{C_\varepsilon} \left[x_i P_{ij} - \left(\frac{1-c}{c}\right)\sigma_{ij}u_i \right] n_j \, \mathrm{d}s$$
$$= \int_{C_\varepsilon} \left[(x_i - x_i^c) P_{ij} - \left(\frac{1-c}{c}\right)\sigma_{ij}u_i \right] n_j ds + x_i^c \int_{C_\varepsilon} P_{ij} n_j \, \mathrm{d}s,$$

where x_i^c is the value of x_i evaluated at the crack tip. The first integral vanishes as $\varepsilon \to 0$; therefore we have

$$\lim_{\varepsilon\to 0} M(C_\varepsilon) = x_i^c \lim_{\varepsilon\to 0} \int_{C_\varepsilon} P_{ij} n_j ds = x_i^c G_{ij} m_j, \tag{8}$$

where eqn (4) has been used in the last step. In the $y_1 y_2$ system of coordinates mentioned above, we have that

$$M(H) = \lim_{\varepsilon\to 0} M(C_\varepsilon) = y_1^c G_{cr}$$

and therefore

$$y_1^c = \frac{M(H)}{G_{cr}}.$$

The inclination of a crack in a two-dimensional problem of hydraulic fracturing, where the fluid is allowed to infiltrate the crack and exert a constant and known pressure on the crack faces, can also be determined using the same technique, since an integral of the term $\sigma_{22}u_{2,1}$ along the crack faces on y_1 can be calculated in terms of the crack opening displacement at the beginning of the crack and the fluid pressure. However, the length of the crack cannot be determined using the M integral.

5. Three-dimensional crack detection

In this section, we consider the problem of determining the orientation in space and area of a planar crack of arbitrary shape which has been initiated from a wellbore. We assume again that there are no stresses at infinity and knowledge of the stress tensor and displacement vector fields are available on the wellbore walls and that the energy release rate per unit of crack front's length g_{cr} is constant and known. We consider two cartesian systems of coordinates $x_1x_2x_3$ and $y_1y_2y_3$ which are positive oriented ($\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$) and positioned as follows: Both systems have their origins at one of the ends of the crack opening on the wellbore wall; x_3 is vertical and pointing upwards; x_1 is horizontal and pointing towards a prescribed direction, for example orthogonal to the wellbore wall if the wellbore is vertical; y_2 joins both ends of the crack opening and y_1 is contained on the crack plane and points away from the wellbore. The orientation of the crack plane with respect to the system of coordinates $x_1x_2x_3$ can be described with two angles α and β which are defined as follows: Let us consider a horizontal planar crack such that both systems of coordinates $x_1x_2x_3$ and $y_1y_2y_3$ coincide, then we rotate the system $y_1y_2y_3$ around the x_1 axis an angle α (following the right hand rule). Thus α gives us the inclination of the crack opening with respect to the horizontal. Finally, we rotate the system $y_1y_2y_3$ an angle β around the y_2 axis; therefore β gives us the deviation of the crack plane with respect to a prescribed direction. This transformation of coordinates can be written as

$$y_i = a_{ij}x_j,$$

where a_{ij} is the cosine of the angle formed by the axes y_i and x_j and, as a function of α and β is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\alpha\sin\beta & -\cos\alpha\sin\beta \\ 0 & \cos\alpha & \sin\alpha \\ \sin\beta & -\sin\alpha\cos\beta & \cos\alpha\cos\beta \end{bmatrix}.$$

Let S_ε be a cylinder of a small radius ε that goes along the crack front L and has a cross section C_ε, then

$$J_i^y(S_\varepsilon) = \int_{S_\varepsilon} P_{ij}^y n_j^y ds \sim \int_L \int_{C_\varepsilon} P_{ij} n_j ds_c ds_L, \tag{9}$$

as $\varepsilon \to 0$.

From the two-dimensional analog (4), we can write that

$$\lim_{\varepsilon\to 0} \int_{C_\varepsilon} P_{ij} n_j ds_c = g_{ij} m_j, \tag{10}$$

where m is a unit vector contained in the crack plane, normal to the crack front and pointing towards the direction of crack advance and g_{ij} is a tensor whose components are: $g_{11} = g_{22} = g_{cr}$ and $g_{12} = -g_{21}$, the other components are not relevant since m is orthogonal to the y_3 axis and J_3^y does not necessarily vanish on the y_1y_2 plane. This form of the tensor g_{ij} holds in any cartesian system of coordinates whose third axis is parallel to y_3. Substituting (10) into (9) we have

$$\lim_{\varepsilon\to 0} J_i^y(S_\varepsilon) = \int_L g_{ij} m_j ds_L.$$

Tracing L anticlockwise (seen from the positive part of y_3), we have that

$$\lim_{\varepsilon\to 0} J_i^y(S_\varepsilon) = g_{i1} \int_L dy_2 - g_{i2} \int_L dy_1 = g_{i1} l, \tag{11}$$

where l is the straight line distance between the two ends of the crack opening. Evaluating (11) for $i = 1$, using the facts that J_1^y is path independent and vanishes on the y_1y_2 plane and expressing it in the $x_1x_2x_3$ coordinates, we arrive at

$$lg_{cr} = J_1^x(H)\cos\beta + J_2^x(H)\sin\beta\sin\alpha - J_3^x(H)\sin\beta\cos\alpha,$$

where H is the wellbore with the normal pointing inwards plus the Earth's surface with the normal pointing upwards. We are again assuming zero stresses at infinity. Since displacements are known around the wellbore, we assume that the numbers l and α can be measured experimentally, therefore the above equation allows us to determine the crack plane's deviation β

$$\beta = \arcsin\left(\frac{lg_{cr}}{\sqrt{J_1^x(H)^2+(J_2^x(H)\sin\alpha-J_3^x(H)\cos\alpha)^2}}\right)-\arctan\left(\frac{J_1^x(H)}{J_2^x(H)\sin\alpha-J_3^x(H)\cos\alpha}\right).$$

In order to determine the area of the crack surface, we consider the M integral. Following a similar argument as that for the J integral, the value of the M integral on a small cylinder S_ε which surrounds the crack front is

$$M(S_\varepsilon)\sim\int_L\int_{C_\varepsilon}\left[y_iP_{ij}^y-\left(\frac{3-2c}{2c}\right)\sigma_{ij}^yu_i^y\right]n_j^y\,ds_c\,ds_L$$

as $\varepsilon\to 0$ and, with a similar analysis to that from which we deduced eqn (8), we have

$$\lim_{\varepsilon\to 0}\int_{C_\varepsilon}\left[y_iP_{ij}^y-\left(\frac{3-2c}{2c}\right)\sigma_{ij}^yu_i^y\right]n_j^y\,ds_c = y_i^cg_{ij}m_j,$$

therefore

$$\lim_{\varepsilon\to 0}M(S_\varepsilon)=\int_L y_ig_{ij}m_j\,ds_L = g_{cr}\int_L(y_1dy_2-y_2dy_1)+g_{21}\int_L(y_1dy_1+y_2dy_2)=2Ag_{cr}+\frac{l^2g_{21}}{2},$$

where the integral along the crack front L is carried out anticlockwise (seen from the positive side of the y_3 axis) and A is the area of the crack surface, which is the area of the section of the crack plane limited by the crack front and a straight line that joins the two ends of the crack opening. The value of g_{21} can be determined from eqn (11). After some arrangements, we find the following formula for the area A

$$A=\frac{M(H)-\dfrac{l}{2}(J_2^x(H)\cos\alpha+J_3^x(H)\sin\alpha)}{2g_{cr}}.$$

6. Conclusion

In this work, a method has been established that allows us to determine the inclination and length of a stress free straight crack in bidimensional problems or the inclination and area of a stress free planar crack of arbitrary shape in three dimensional problems. The method is valid for some non-linear materials (the J integral formulation is valid for any energy density function and that involving the M integral for stress-strain laws whose Lagrangian is a homogeneous function of some degree in the strain components) and mixed mode loading and requires knowledge of stresses and displacements on the body surface (the wellbore and Earth's surface) and the value of the critical energy release rate per unit of crack front's length necessary for the crack to propagate.

This method has special applications in hydraulic fracturing but can also be used in other kind of problems like pressure vessels, etc.

The technique is based on the application of two types of path independent integral that are derived from variational principles. The path independent integrals used in this method vanish along the stress free crack and are supposed to be completely determined on the body surface. This means that the complete stress tensor and displacement vector fields must be known on the body surface. This is possible if both surface tractions and displacements are known and numerical differentiation of the displacements is permitted within a reasonable degree of accuracy.

In the particular case of hydraulic fracturing, the method presented in this work can be used to determine the inclination and area of stress free planar crack. Since the crack is assumed to be stress free, the fluid must not be allowed to flow inside the crack. This could be achieved by means of a technique called sleeve fracturing. The material can be modeled as a non-linear elastoplastic material of a power law type. The displacement inside the wellbore can be measured by means of a caliper tool and the value of the stress tensor and displacement vector fields on the Earth's surface can be taken from an analytic unfractured model (a pressurized vertical wellbore in a stress free half space for example) under the principle that the effects of the fracture are negligible on the Earth's surface.

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